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Inverse problems for non-uniformly polarized piezoelectric rods☆

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Abstract

Formulations of inverse coefficient problems of electroelasticity for non-hyperbolic-type operators are presented and calculation schemes for determining the piezoelectric modulus as a function of the coordinates in the one-dimensional case are proposed. The results of computer simulation are presented.

The operation of some piezoelectric elements is based on a change in the polarization of a piezoelectric ceramics under the action of an electric field, a mechanical load and the temperature. For example, partial depolarization of a piezoelectric element can occur when part of is heated above the Curie point,[1] and a change in the effective piezoelectric modulus may be connected with the partial removal of the electrodes from the surface of the piezoelectric element. Modelling of the depolarization phenomenon in electroelasticity is one of the pressing problems, for the solution of which several different approaches have been proposed.

Below, partial depolarization is modelled by postulating a non-uniformity of the piezoelectric properties, assuming the moduli of elasiticity and the permittivities are constant[2,3] within the framework of the model of linear electroelasticity. The problem of determining the degree of depolarization is reduced to determining the dependence of the piezoelectric moduli on the coordinates from known functionals or operators of the solutions. Either the amplitude dependences of the current in the circuit or the displacements of part of the surface under the mechanical or electrical loading are used as such information. In a number of papers (see Refs. 4–6, etc.) in different formulations of the piezoelectric modulus, the electroelasticities (either coefficient or boundary) were determined by solving a certain inverse problem.

At the present time, a fair amount of experience has been acquired in investigating one-dimensional inverse coefficient problems for hyperbolic-type operators. The methods proposed are based either on first reducing the problems to a non-linear Volterra-type operator equation,[7–9] or by the direct use of methods of inverting a difference scheme.[10] The non-hyperbolic form of the electroelasticity operator means that it is not possible to use these approaches directly and new methods for solving inverse problems are required.

1. Formulation of the problem

Consider the oscillations of a non-uniform electroelastic body V, bounded by a surface, part of which is covered with infinitely thin electrodes S_+ and S_-; the piezoelectric properties here differ in the different subregions. Using the

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model of linear electroelasticity, we will assume that the piezoelectric moduli are functions of the coordinates, while the elastic moduli and the permittivities are constant in the volume V, which enables us to model the weakening of the piezoelectric properties (partial or complete depolarization) of the piezoelectric element.

The constitutive relations for non-uniform piezoelectric bodies have the form[1]

$$u_{i,j} = s_{ijkl}\sigma_{kl} - d_{ijk}(\mathbf{x})\varphi_{,k}, \quad D_k = d_{kij}(\mathbf{x})\sigma_{ij} - \ni_{kl}\varphi_{,l} \tag{1.1}$$

where s_{ijkl} are the components of the elastic compliance tensor, $\ni_{kl}$ are the components of the permittivity tensor, and $d_{ijk}(\mathbf{x})$ are the piezoelectric moduli, where $d_{ijk}(\mathbf{x}) = 0$ in the region of complete depolarization. Note that the simplest method of describing depolarization is the model in which the weakening of the piezoelectric properties occurs abruptly in the polarization region.

The complete system of equations for a linear piezoelectric medium, ignoring temperature effects, is obtained by substituting expressions (1.1) into the equations of motion $\sigma_{ij,j} = \rho\ddot{u}_i$ and the equation of electrostatics $D_{i,i} = 0$.

We will assume that oscillations are excited by applying a potential difference to the electrodes S_+ and S_-. The boundary conditions can then be represented in the following form (they correspond to electric loading of the body)

$$\sigma_{ij}n_j|_S = 0, \quad D_kn_k|_{S_H} = 0, \quad \varphi|_{S_\pm} = \pm V(t) = \pm V_0\psi(t) \tag{1.2}$$

where n_k are the components of the unit vector of the outward normal to S, and $S = S_H \cup S_+ \cup S_-$, S_H is the region not covered by an electrode. We will assume that the initial conditions are zero. We will formulate the inverse problem of determining the piezoelectric moduli $d_{ijk}(\mathbf{x})$ (or some of them) as functions of $L_2(V)$ using certain additional information on the displacement fields on part of the boundary or the current in the circuit in which the piezoelectric element is connected.

This general formulation leads to a multidimensional inverse coefficient problem for the non-hyperbolic operator of electroelasticity, which has not been investigated in the literature. In order to simplify the formulation, we will consider the case when the region V is a cylindrical body, the transverse dimensions of which are considerably less than the longitudinal dimension, i.e. we will consider a rod model, which is often used in experiments and serves to determine the piezoelectric constants. To reduce the inverse problem to an operator equation, it is necessary to construct a solution of the direct problem in explicit form for an arbitrary depolarization law. Note that, in certain formulations the differential equations of the initial-boundary-value problem, describing the oscillations of the rod, have variable coefficients, and it is impossible to construct its solution in explicit form. This problem can be solved by reducing the initial-boundary-value problem to a Fredholm integral equation of the second kind in Laplace-transform space. This approach works when the information regarding the boundary fields or the current is available over an infinite time interval.

Thus, we will further consider a piezoelectric rod of length $2l$, situated along the x_1 axis, with electrodes deposited on the surface perpendicular to the x_3 axis (Fig. 1). Oscillations of the rod are caused by applying a potential difference $2V(t)$ to its electroded parts.

Assuming further that the surface of the rod is free from mechanical stresses and the initial conditions are non-zero, we will consider several different formulations of the inverse problems of determining the piezoelectric moduli in the case of longitudinal and transverse polarization of the rod for known information on the displacement of its ends or the current in the circuit.

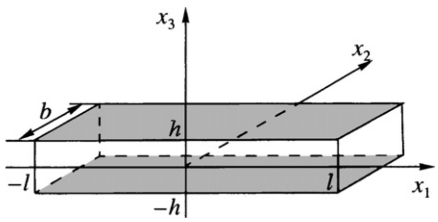

Fig. 1.

2. Determination of the piezoelectric moduli from the displacement of the ends of the rod in the case of transverse polarization

We will consider the case when the dimension of the rod $2l$ is much greater than its dimensions $2h$ and b. The problem of determining the electromechanical fields in the rod will then be one-dimensional, and the dependence of these quantities on the coordinates x_2 and x_3 can be neglected. Moreover, we can take all the components of the stress tensor, apart from σ_{11}, to be equal to zero. The application of a potential difference to the electrodes is equivalent to producing, in the whole volume of the rod, an electric field of the form

$$E_1 = E_2 = 0, \quad E_3 = -\varphi_{,3} = E(t) = -V(t)/(2h)$$

In this case, the constitutive relations can be written in the form

$$u_{1,1} = s_{11}\sigma_{11} + d_{31}(x_1)E(t), \quad D_3 = d_{31}(x_1)\sigma_{11} + э_{33}E(t), \quad D_1 = 0 \tag{2.1}$$

Substituting the expression for σ_{11} from the first equation of (2.1) into the equation of motion $\sigma_{1,1} = \rho\ddot{u}_1$, we will have the following initial-boundary-value problem for finding $u_1(x_1, t)$

$$u_{1,11} - d_{31,3}(x_3)E(t) = c^{-2}\ddot{u}_1, \quad u_{1,1}|_{x_1 = \pm l} = d_{31}(\pm l)E(t); \quad c^{-2} = \rho s_{11} \tag{2.2}$$

Note that the equations of electrostatics in this case are satisfied automatically.

We will formulate the inverse problem of determining the function $d_{31}(x_1)$ from known information on the relative displacement of the ends of the rod

$$u_1(l, t) - u_1(-l, t) = f(t), \quad t \geq 0 \tag{2.3}$$

At the first stage, we will construct the solution of the direct problem (2.2) using a Laplace transformation with respect to time. Taking the boundary and initial conditions into account and changing to dimensionless variables

$$x_1 = l(2y-1), \quad q(y) = \frac{d_{31}(l(2y-1))}{d_{31}(-l)}$$
$$u(y, p) = \frac{\tilde{u}_1(l(2y-1))}{2l\tilde{E}(p)d_{31}(-l)}, \quad \Omega^2 = 4l^2\frac{p^2}{c^2} \tag{2.4}$$

where

$$\tilde{u}_1(x_1, p) = \int_0^\infty u_1(x_1, t)e^{-pt}dt, \quad \tilde{E}(p) = \int_0^\infty E(t)e^{-pt}dt$$

we obtain the following boundary-value problem in Laplace transformation space (obviously $q(0) = 1$)

$$u''(y, \Omega) - \Omega^2 u(y, \Omega) = q'(y) \tag{2.5}$$

$$u'(y, \Omega)|_{y=0} = 1, \quad u'(y, \Omega)|_{y=1} = q(1) \tag{2.6}$$

Here the unknown function $q(y)$, representing the non-uniformity of the polarization, occurs on the right-hand side of differential Eq. (2.5) with constant coefficients, and its solution can be found in explicit form

$$u(y, \Omega) = A(p)\mathrm{sh}\Omega y + B(p)\mathrm{ch}\Omega y + \Phi(y, p) \tag{2.7}$$

where

$$\Phi(y, p) = \frac{1}{2}\int_0^1 \mathrm{sgn}(y-\xi)q(\xi)e^{-\Omega|y-\xi|}d\xi \tag{2.8}$$

By satisfying the boundary conditions, we obtain the solution of problem (2.5), (2.6) in the form

$$u(y,\Omega) = \frac{1}{2}\int_0^1 q(\xi)G(y,\Omega,\xi)d\xi$$
$$G(y,\Omega,\xi) = \frac{1}{\mathrm{sh}\Omega}(e^{-\Omega(1-\xi)}\mathrm{ch}\Omega y - e^{-\Omega\xi}\mathrm{ch}\Omega(1-y)) + \mathrm{sgn}(y-\xi)e^{-\Omega|y-\xi|} \tag{2.9}$$

Taking relations (2.3) and (2.9) into account, we obtain a linear operator equation with a smooth kernel, connecting the unknown function $q(y)$ and the known functions $\tilde{f}(\Omega)$ and $\tilde{E}(\Omega)$

$$\int_0^1 q(y)K(y,\Omega)dy = \tilde{F}(\Omega), \quad \Omega \in [0,\infty) \tag{2.10}$$

where

$$K(y,\Omega) = \mathrm{ch}\Omega(1-2y), \quad \tilde{F}(\Omega) = -\frac{\tilde{f}(\Omega)}{4l\tilde{E}(\Omega)d_{31}(-l)}$$

Hence, the inverse problem of determining the function $d_{31}(x_1)$ from known information on the relative displacement of the ends of the rod reduces to solving a Fredholm integral equation of the first kind with a smooth kernel of the form (2.10) and is a typical ill-posed problem.[11]

3. Determination of the piezoelectric modulus from the current in the circuit for longitudinal polarization

We will now consider the case when $2h \gg 2l$, $2h \gg b$. We can then only assume σ_{33} to be non-zero, and the constitutive relations (1.1) are written in the form

$$u_{3,3} = s_{33}\sigma_{33} - d_{33}(x_3)\varphi_{,3}, \quad D_3 = d_{33}(x_3)\sigma_{33} - \ni_{33}\varphi_{,3} \tag{3.1}$$

Of the equations of motion for an electroelastic medium two remain here, namely,

$$\sigma_{33,3} = \rho\frac{\partial^2 u}{\partial t^2}, \quad D_{3,3} = 0 \tag{3.2}$$

Note that, usually in electroelasticity problems we set up a system in terms of the variables u_3 and φ, but in the problem considered here it is more convenient to formulate it in terms of the unknowns σ_{33} and φ. Eliminating the displacements from relations (3.1) and (3.2), we obtain

$$\sigma_{33,33} = \rho\frac{\partial^2}{\partial t^2}(s_{33}\sigma_{33} - d_{33}(x_3)\varphi_{,3}), \quad (d_{33}(x_3)\sigma_{33} - \ni_{33}\varphi_{,3})_{,3} = 0$$
$$\sigma_{33}|_{x_3=\pm h} = 0, \quad \varphi|_{x_3=\pm h} = \pm V(t) \tag{3.3}$$

By virtue of the dependence of the piezoelectric modulus on the longitudinal coordinate, analytical methods of constructing a solution of problem (3.3) are not possible for arbitrary functions $d_{33}(x_3)$, and we will therefore reduce it to a Fredholm integral equation of the second kind. To do this we apply a Laplace transformation with respect to time to problem (3.3) and, eliminating the potential $\tilde{\varphi}$, we obtain the following expression for the dimensionless stress $Y(y, \Omega)$

$$Y(y,\Omega) = \Omega^2\int_0^1 K(y,\xi)[(1-k_0^2q^2(\xi))Y(\xi,\Omega) - q(\xi)]d\xi \tag{3.4}$$

Here

$$K(y) = \begin{cases} (y-1)\xi, & y > \xi \\ y(\xi - 1), & y < \xi \end{cases}$$

and we have introduced the dimensionless quantities

$$y = \frac{x_3 + h}{2h}, \quad q(y) = \frac{d_{33}(h(2y-1))}{d_{33}(-h)}, \quad \Omega^2 = 4h^2p^2\rho s_{33}$$
$$Y(y, \Omega) = \frac{\tilde{\sigma}_{33}(h(2y-1), \Omega)s_{33}}{\tilde{E}(\Omega)d_{33}(-h)}, \quad k_0^2 = \frac{d_{33}^2(-h)}{\ni_{33}s_{33}} \tag{3.5}$$

Integral Eq. (3.4) can be solved numerically by using, for example, quadrature formulae and reducing it to a linear algebraic system in terms of the node unknowns.

Suppose, as additional information for solving the inverse coefficient problem of determining the degree of depolarization of the rod, we assume that we know the current in the circuit in which the piezoelectric element is connected, namely,

$$-\ni_{33}S_0\frac{\partial}{\partial t}\varphi_{,3}\Big|_{x_3 = h} = I(t) \tag{3.6}$$

where S_0 is the area of the electroded surface. Integrating this equality with respect to t, we obtain

$$\varphi_{,3}|_{x_3 = h} = -\ni_{33}S_0\int_0^t I(\tau)d\tau = E^*(t) \tag{3.7}$$

where $E^*(t)$ is a known differentiable function and $E^*(0) = 0$.

Hence, the inverse problem consists of finding the piezoelectric modulus $d_{33}(x_3)$, $x_3 \in [-h, h]$ for known functions $V(t)$ and $E^*(t)$, $t \geq 0$. Using a Laplace transformation, we can reduce this problem to the following system of non-linear operator equations in the functions $Y(y, \Omega)$ and $q(y)$

$$Y(y, \Omega) = \Omega^2\int_0^1 K(y, \xi)[(1 - k_0^2q^2(\xi))Y(\xi, p) - q(\xi)]d\xi = K_1(q, Y) \tag{3.8}$$

$$K_2(q, Y) = \int_0^1 q(y)Y(y, \Omega)dy = \tilde{F}(\Omega) \tag{3.9}$$

where

$$\tilde{F}(\Omega) = \frac{1}{k_0^2}\left(\frac{1}{h}\frac{\tilde{V}(\Omega)}{\tilde{E}^*(\Omega)} - 1\right)$$

We will construct an iteration process for finding $q(y)$ according to the following rule

$$Y^{(n)} = K_1(q^{(n)}, Y^{(n)}), \quad \tilde{F} = K_2(q^{(n+1)}, Y^{(n)}); \quad n = 0, 1, 2, \ldots \tag{3.10}$$

Note that here the first equation is a linear Fredholm integral equation of the second kind in $Y^{(n)}$ and can be solved by the standard method. The second equation, for the known function $Y^{(n)}$, is a Fredholm integral equation of the second kind in $q^{(n+1)}$ and requires regularization for the solution. Assuming $q^{(n+1)} = q^{(n)} + \eta$, we obtain at each iteration a system of linear integral equations in $Y^{(n)}(y, \Omega)$, and corrections $\eta(y)$ of the following form

$$Y^{(n)} = K_1(q^{(n)}, Y^{(n)}), \quad \tilde{F} - K_2(q^{(n)}, Y^{(n)}) = K_2(\eta, Y^{(n)}) \tag{3.11}$$

This system can be solved numerically by a combination of a finite-dimensional approximation of integral operators and the method of regularization.Note that this formulation has a serious drawback, namely, the kernel of the integral operator $K_2(q, Y)$ vanishes at the ends of the section of integration, which, generally speaking, leads to a considerable error in determining the function $q(y)$ in the neighbourhood of the point $y=1$ (it is obvious that $q(0)=1$). When information is available on the value of the function $q(y)$ at the point $y=1$, we would expect a smaller error when solving the inverse problem, and we can choose as the initial approximation $q^{(0)}$ of iteration scheme (3.11) a function which satisfies the conditions

$$q^{(0)}(0) = q(0) = 1, \quad q^{(0)}(1) = q(1)$$

4. Determination of the piezoelectric modulus from known information on the displacement of the ends of the rod for longitudinal polarization

The oscillations of a piezoelectric rod, polarised along the d_3 axis, due to the action of a potential difference $2V(t)$, applied to the electroded parts of the rod, are described by a system of equations and boundary conditions of the form (3.3). We will assume, as additional information for determining the piezoelectric modulus $d_{33}(x_3)$, the known relative displacement of the ends of the rod

$$u_1(l, t) - u_1(-l, t) = f(t), \quad t \ge 0 \tag{4.1}$$

We will introduce the following notation

$$M_1(\Omega) = 1 + k_0^2\int_0^1 q(y)Y(y, \Omega)dy, \quad M_2(\Omega) = \int_0^1 [(1 - k_0^2q^2(y))Y(y, \Omega) - q(y)]dy \tag{4.2}$$

As in the previous case, after a Laplace transformation with respect to time the direct problem reduces to a Fredholm integral equation of the second kind of the form (3.8); the function $\tilde{E}(\Omega)$ in this case is found from the electrical boundary conditions

$$\tilde{E}(\Omega) = l^{-1}\tilde{V}(\Omega)/M_1(\Omega)$$

We determine the transform of the relative displacement of the ends

$$\tilde{F}(\Omega) = \frac{\tilde{u}(l, \Omega) - \tilde{u}(-l, \Omega)}{2d_{33}(-l)\tilde{V}(\Omega)} = K_2(q, Y), \quad K_2(q, Y) = \frac{M_2(\Omega)}{M_1(\Omega)} \tag{4.3}$$

Hence, the problem of determining the depolarization of the rod reduces to a system of operator equations of the form (3.8), (4.3). We will construct an iteration scheme for solving system (3.8), (4.3) by representing it in the form (3.10). Whereas in the previous case the second of the integral operators of the system was linear, in this case the operator $K_2(q, Y)$ is non-linear in $q(y)$, and the second equation of (3.10) is solved by the linearization method in the neighbourhood of $q^{(n)}$. In this approach we assume $q^{(n+1)} = q^{(n)} + \eta$, and for the function η we obtain the following linear Fredholm integral equation of the first kind with smooth kernel

$$\tilde{F} - K_2(q^{(n)}, Y^{(n)}) = K_2'(\eta, Y^{(n)}) \tag{4.4}$$

where $K_2'(\eta, Y^{(n)})$ is the Frechét derivative with respect to η of the operator $K_2(\eta, Y^{(n)})$

$$K_2'(\eta, Y^{(n)}) = \int_0^1 L(\xi, \Omega)\eta(\xi)d\xi$$

$$L(\xi, \Omega) = -\frac{M_1^{(n)}(\Omega)[2k_0^2q^{(n)}(\xi)Y^{(n)}(\xi, \Omega) + 1] + M_2^{(n)}(\Omega)k_0^2Y^{(n)}(\xi, \Omega)}{M_1^{(n)^2}(\Omega)}$$

The functions $M_1^{(n)}(\Omega)$, $M_2^{(n)}(\Omega)$ are defined by relations (4.2) by replacing the functions $q(y)$ and $Y(y, \Omega)$ by $q^{(n)}(y)$, $Y^{(n)}(y, \Omega)$ in them.

Hence, the solution of the initial inverse problem reduces to solving a system of linear integral equations at each step of the iteration process

$$q^{(0)} = 1, \quad Y^{(n)} = K_1(q^{(n)}, Y^{(n)}), \quad \tilde{F} - K_2(q^{(n)}, Y^{(n)}) = K_2'(\eta, Y^{(n)}) \tag{4.5}$$

which can also be solved numerically in the same way as system (3.11).

5. Computer simulation

We will consider a number of model examples of establishing the piezoelectric modulus using the schemes described above.

To solve Eq. (2.10) and systems (3.11) and (4.5) we will first map $\Omega \in [0, \infty) \to \zeta \in [0, 1]$, making the replacement

$$\varsigma = \frac{\Omega^2}{1+\Omega^2}, \quad \Rightarrow \Omega = \sqrt{\frac{\varsigma}{1-\varsigma}}, \quad \varsigma \in [0, 1] \tag{5.1}$$

We will approximate the integrals in the integral operators on a uniform grid $\{y_j\}_{j=1}^{n}$ in the interval [0,1] for each of the values of the argument ζ on a uniform grid $\{\zeta_i\}_{i=1}^{m}$ in the interval [0,1]. Hence, the linear integral equations are reduced to systems of linear algebraic equations, which, in view of the fact that the original problem is ill-posed, will also be ill-posed and require special regularizing methods for inversion.

Using the proposed schemes to model the procedure for solving inverse problems, we carried out a number of computer simulations for different positive functions $q(y)$, monotonically non-increasing in the interval [0,1]. When specifying the form of these functions (linear, quadratic, exponential and step), the input data is determined in accordance with relations (2.3), (3.7) and (4.1). The inverse problem is then solved in one of the formulations described above.

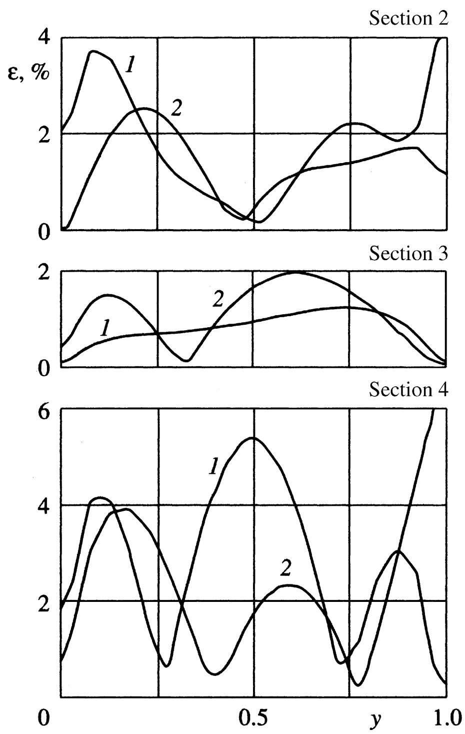

Fig. 2.

We will present the results of computer simulation for the following laws

$$1)\ q(y) = 1 - y^2/2,\ 2)\ q(y) = e^{-2y}$$

which model, for example, the situation when one end of the rod is subjected to a heavy load. We will use Tikhonov's method[11] as the regularizing algorithm.

The error ε in establishing the unknown function $q(y)$ from the inverse coefficient problems presented in Sections 2, 3 and 4 is shown in Fig. 2. Curves 1 and 2 correspond to the chosen laws 1 and 2.

For the problems from Sections 2 and 4 we chose the function $q^{(0)} = 1$ as the initial approximation, while for the problem from Section 3 we chose a function of the form $q^{(0)}(y) = q(1) - q(0))y + q(0)$; attempts to use the function $q^{(0)} = 1$ as the initial approximation in this problem led to considerable error in establishing the function $q(y)$ (about 20%).

An analysis of the results of the computer simulation confirms that the error in establishing the piezoelectric modulus does not exceed 7% when using the iteration method, even with a small number of iterations (from 2 to 5); further calculations using the iteration scheme hardly improve the result. Note also that the number of points of subdivision has no particular effect on the reconstruction results when reducing the problems to systems of linear algebraic equations.

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